

A NOTE ON QUANTUM GEOMETRIC LANGLANDS DUALITY, GAUGE THEORY, AND QUANTIZATION OF THE MODULI SPACE OF FLAT CONNECTIONS

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ABSTRACT. Montonen-Olive duality implies that the categories of A-branes on the moduli spaces of Higgs bundles on a Riemann surface C for groups G and ${}^L G$ are equivalent. We reformulate this as a statement about categories of B-branes on the quantized moduli spaces of flat connections for groups $G_{\mathbb{C}}$ and ${}^L G_{\mathbb{C}}$. We show that it implies the statement of the Quantum Geometric Langlands duality with a purely imaginary “quantum parameter” which is proportional to the inverse of the Planck constant of the gauge theory. The ramified version of the story is also considered.

1. INTRODUCTION

Langlands duality has many different manifestations in mathematics and physics. The most famous examples in mathematics are the Langlands program for number fields and function fields and its geometric counterpart. It has been proposed a long time ago by M. Atiyah that Langlands Program is related to Montonen-Olive duality in gauge theory. Montonen-Olive duality [13] (formulated more precisely by Osborn [14]) is a conjecture that four-dimensional $N = 4$ supersymmetric gauge theories with gauge groups G and ${}^L G$ are isomorphic if their coupling constants are inversely related. Here G and ${}^L G$ are compact simple Lie groups such that the character lattice of G is isomorphic to the cocharacter lattice of ${}^L G$ and vice versa. In [12] it was shown how the main statements of the Geometric Langlands Program in the unramified case can be deduced by considering a topologically twisted version of the Montonen-Olive duality. This was later extended to the ramified case [7, 18, 8].

Geometric Langlands duality has a “baby” version which says that the derived categories of coherent sheaves of the moduli spaces of Higgs bundles over a Riemann surface C for a group G and its Langlands-dual ${}^L G$ are equivalent. This statement can be deduced by applying the Montonen-Olive duality to a holomorphic-topological twist of the $N = 4$ supersymmetric gauge theory [11].

Yet another mathematical manifestation of Langlands duality is the Quantum Geometric Langlands (QGL) duality which relates the categories of twisted D-modules over moduli spaces (actually, stacks) of principal $G_{\mathbb{C}}$ and ${}^L G_{\mathbb{C}}$ bundles over C . We denote these moduli spaces $\mathrm{Bun}_G(C)$ and $\mathrm{Bun}_{^L G}(C)$. QGL is a generalization of the usual Geometric Langlands duality, in the sense that in a certain “classical” limit the former reduces to the latter.¹ For a review of QGL and its relation to Conformal Field Theory see [3]; for some physical manifestations of QGL see [16, 6].

A twisted D-module over a complex manifold M is a module over the sheaf of (holomorphic) differential operators on a complex power of a holomorphic line bundle \mathcal{L} over M . In the QGL context \mathcal{L} is a certain line bundle \mathcal{L}_0 over $\mathrm{Bun}_G(C)$,² so twisted D-modules are modules over the algebra of differential operators on $\mathcal{L}_0^{q-h^\vee}$, $q \in \mathbb{C}$. Here

¹Confusingly, the “baby” version of the Geometric Langlands duality mentioned in the previous paragraph is sometimes referred to as the classical limit of the Geometric Langlands duality. We will not use this terminology in this paper.

²For $G = SU(N)$ \mathcal{L}_0 is the determinant line bundle; in general it is determined by the property that its first Chern class generates the second cohomology of $\mathrm{Bun}_G(C)$.

h^\vee is the dual Coxeter number of G . Note that $\mathbf{q} = 0$ corresponds not to untwisted D-modules, but to modules over holomorphic differential operators on the line bundle $\mathcal{L}_0^{-h^\vee}$; this shift in parametrization is introduced for future convenience. We will refer to the twist by $\mathcal{L}_0^{-h^\vee}$ as the critical twist.

According to QGL, the derived categories of twisted D-modules on $\mathrm{Bun}_G(C)$ and $\mathrm{Bun}_{^LG}(C)$ are equivalent if the twist parameters \mathbf{q} and ${}^L\mathbf{q}$ are related by

$$(1) \quad {}^L\mathbf{q} = -\frac{1}{n_{\mathfrak{g}} \mathbf{q}},$$

where $n_{\mathfrak{g}} = 1, 2$, or 3 depending on the maximal multiplicity of edges in the Dynkin diagram of G . If we interpret \mathbf{q} as the inverse of a “Planck constant”, then the “classical” limit $\mathbf{q} \rightarrow \infty$ on one side is dual to the “ultra-quantum” limit ${}^L\mathbf{q} \rightarrow 0$ on the other side. (We will see shortly why it is natural to identify \mathbf{q} as the inverse of the Planck constant.) In this limit QGL duality reduces to classical geometric Langlands duality. Namely, one gets the statement that the derived category of coherent sheaves on the moduli space of flat $G_{\mathbb{C}}$ connections (which is regarded as the $\mathbf{q} \rightarrow \infty$ limit of the derived category of twisted D-modules on $\mathrm{Bun}_G(C)$) is equivalent to the derived category of critically twisted D-modules on $\mathrm{Bun}_{^LG}(C)$.

In [12] QGL duality was interpreted in terms of equivalence of categories of A-branes on the Hitchin moduli spaces $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$ with respect to a certain complex structure K and with a nonzero B-field. The B-field is determined by a gauge theory parameter θ which takes values in \mathbb{R} . It was argued in [12] that the category of A-branes on $\mathcal{M}_H(G, C)_K$ is equivalent to the category of twisted D-modules on $\mathrm{Bun}_G(C)$ with the twist parameter

$$\mathbf{q} = \frac{\theta}{2\pi}.$$

Montonen-Olive duality implies that the categories of A-branes for G and LG are equivalent if \mathbf{q} and ${}^L\mathbf{q}$ are related as in (1); this implies the statement of the QGL duality for real \mathbf{q} .

One drawback of this physical derivation is that \mathbf{q} is naturally real, while in QGL it is complex. Another somewhat unsatisfactory feature is that from the physical viewpoint θ is unrelated to the Planck constant of the underlying gauge theory; the latter does not appear because it affects only the BRST exact terms in the gauge theory action.

In this note we describe a variation of the argument of [12] which provides an alternative derivation of the QGL duality from gauge theory.

Our starting point will again be the dual pair of $N = 4$ supersymmetric gauge theories with gauge groups G and ${}^L G$. We will twist them into a dual pair of topological field theories as in [12]. Recall that each of these topological field theories has a 1-parameter family of BRST operators labeled by $t \in \mathbb{C} \cup \{\infty\}$. The Montonen-Olive duality acts on this family by

$$t \rightarrow {}^L t = t \frac{|\tau|}{\tau},$$

where τ is the complexified gauge coupling of the $N = 4$ gauge theory:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

Classical Geometric Langlands duality is related to the special case $\theta = 0$, $t = i$, ${}^L t = 1$. In this note we look at the case $t = {}^L t = 0$, with arbitrary θ and e^2 . It will be shown below that for $\theta = 0$ the relevant category is the derived category of twisted D-modules on $\mathrm{Bun}_G(C)$ with $q = \bar{\tau}$. Thus the parameter q is now purely imaginary, and its imaginary part is precisely the inverse of the gauge-theory Planck constant. The limit $q \rightarrow \infty$ is the classical limit in the gauge theory, so classical Geometric Langlands duality emerges as one of the two dual theories becomes classical.

The interpretation in terms of branes also changes: instead of A-branes in complex structure K , we will obtain A-branes in complex structure I , with respect to which $\mathcal{M}_H(G, C)$ can be identified with the moduli space of Higgs bundles on C . This interpretation makes sense for arbitrary θ . The connection with twisted D-modules arises through an analogue of the canonical coisotropic brane of [12]; unlike in [12], this analogue exists only for $\theta = 0$. As explained below, yet another reformulation of the QGL is as a statement of derived-Morita-equivalence between the quantized moduli spaces of flat connections for $G_{\mathbb{C}}$ and ${}^L G_{\mathbb{C}}$.

It is interesting to note that Goncharov and Fock [5] considered the quantization of certain moduli spaces related to the moduli space of flat connections and found signs of Langlands duality. However, the details of their set-up are rather different from ours and we do not understand the precise relationship.

An important role in the Geometric Langlands duality is played by Wilson and 't Hooft line operators. For $t = 0$ and $t = \infty$ we argue that there are no such line operators in the topologically twisted gauge theory. On the other hand, there are still interesting surface operators defined in [7], as well as line operators bound to surface operators.

We discuss how the inclusion of surface operators leads to a ramified version of Quantum Geometric Langlands duality.

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2. GL-TWISTED THEORY AT $t = 0$

We use the notation of [12] and [7] throughout. The fields of the GL-twisted theory are a connection A on a principal G -bundle E , a 1-form ϕ with values in $\text{ad}(E)$, a section σ of $\text{ad}(E)_{\mathbb{C}}$, a pair of 1-forms $\psi, \tilde{\psi}$ with values in $\text{ad}(E)_{\mathbb{C}}$, a pair of sections $\eta, \tilde{\eta}$ of $\text{ad}(E)_{\mathbb{C}}$, and a 2-form χ with values in $\text{ad}(E)_{\mathbb{C}}$. The fields A, ϕ, σ are bosonic, while $\psi, \tilde{\psi}, \eta, \tilde{\eta}, \chi$ are fermionic. As in [12], we use the convention that the covariant derivative is $d + A$; thus in a unitary trivialization A and ϕ are anti-Hermitian.

The GL-twisted theory has two supercommuting BRST operators Q_ℓ and Q_r originating from the left-handed and right-handed supersymmetries of the $N = 4$ gauge theory. The most general BRST operator is therefore

$$Q_t = Q_\ell + tQ_r, \quad t \in \mathbb{C} \cup \{\infty\}.$$

In this paper we will study the topological theory at $t = 0$. This is the case when the BRST operator is "chiral", i.e. is purely left-handed. The BRST transformations take the form

$$\begin{aligned} \delta A &= \psi, & \delta \phi &= -i\tilde{\psi}, \\ \delta \sigma &= 0, & \delta \bar{\sigma} &= i\eta, \\ \delta \eta &= [\bar{\sigma}, \sigma], & \delta \tilde{\eta} &= -D * \phi, \\ \delta \psi &= D\sigma, & \delta \tilde{\psi} &= -[\phi, \sigma], \\ \delta \chi &= (F - \phi \wedge \phi)^+ - (D\phi)^-. \end{aligned}$$

The theory has a $U(1)$ symmetry (ghost-number symmetry) with respect to which σ has charge 2, $\psi, \tilde{\psi}$ have charge 1, $\eta, \tilde{\eta}, \chi$ have charge -1 , and A, ϕ have charge 0.

The only local BRST-invariant and gauge-invariant observables in the theory are gauge-invariant polynomials built out of σ . Since they have positive ghost-number charge, all correlators of these operators

vanish. We will see below the theory has no interesting line operators, but it has surface operators as well as a variety of boundary conditions.

On a compact manifold without boundary and without insertion of nonlocal operators, one may consider computing the partition function. The action can be written in the form

$$S = \{Q, V\} + \frac{i\bar{\tau}}{4\pi} \int_M \text{Tr } F \wedge F,$$

where $\bar{\tau}$ is the complex-conjugate of

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

Thus the partition function is an anti-holomorphic function of τ . The path-integral localizes on configurations of A, ϕ, σ such that

- (2) $(F - \phi \wedge \phi)^+ = 0, \quad (D\phi)^- = 0,$
- (3) $D\sigma = 0, \quad [\sigma, \bar{\sigma}] = 0,$
- (4) $[\phi, \sigma] = 0,$

One obvious class of solutions of these equations is given by anti-self-dual connections (instantons) with $\phi = \sigma = 0$. However, the instanton contribution to the partition function vanishes because of fermionic zero modes, and in fact the partition function is independent of τ . Indeed, the partition function is independent of t , we can use any of the BRST-operators Q_t to compute it, and the partition function for $t = i$ is τ -independent, as explained in [12].

3. REDUCTION TO TWO DIMENSIONS

Consider now the GL-twisted theory on a manifold of the form $C \times \Sigma$, where C and Σ are Riemann surfaces. As explained in [12], for $t = 0$ the effective field theory on Σ is the A-model whose target is the moduli space of Higgs bundles on C , which we denote $\mathcal{M}_H(G, C)$. The A-model depends only on the Kähler form ω and B-field B on $\mathcal{M}_H(G, C)$ which are given by

$$B + i\omega = -\bar{\tau}\omega_I,$$

where

$$\omega_I = -\frac{1}{4\pi} \int_C \text{Tr} (\delta A \wedge \delta A - \delta\phi \wedge \delta\phi).$$

Note that ω_I is independent of the complex structure on C ; this is a reflection of the fact that the GL-twisted theory is a topological field theory on $M = C \times \Sigma$.

As discussed in [12], the moduli space $\mathcal{M}_H(G, C)$ is hyperkähler, so it has three complex structures I, J, K satisfying $IJ = K$. The form ω_I is Kähler with respect to I . The other two Kähler forms are

$$\begin{aligned}\omega_J &= \frac{i}{4\pi} \int_C dz d\bar{z} \text{Tr} (\delta\phi_{\bar{z}} \wedge \delta A_z + \delta\phi_z \wedge \delta A_{\bar{z}}), \\ \omega_K &= \frac{1}{2\pi} \int_C \text{Tr} \delta\phi \wedge \delta A\end{aligned}$$

The form ω_J depends on the choice of complex structure on C , while ω_K obviously does not, just like ω_I . On the other hand, both ω_J and ω_K are exact, while ω_I is not; we normalized it so that its periods are integer multiples of 2π . In fact, the second cohomology group of $\mathcal{M}_H(G, C)$ is \mathbb{Z} , and the cohomology class of $\omega_I/2\pi$ generates it.

We note for future use that the form ω_I is cohomologous to

$$(5) \quad -\frac{1}{4\pi} \int_C \text{Tr} \delta A \wedge \delta A = -\frac{1}{2\pi} \int_C dz d\bar{z} \text{Tr} \delta A_z \wedge \delta A_{\bar{z}}.$$

This form is a pull-back of a $(1, 1)$ form on $\text{Bun}_G(C)$ whose periods are integer multiples of 2π and therefore can be thought of as the curvature of a certain holomorphic line bundle \mathcal{L}_0 over $\text{Bun}_G(C)$. For $G = SU(N)$ this is simply the determinant line bundle.

4. FROM A-BRANES TO NONCOMMUTATIVE B-BRANES

The Montonen-Olive duality of $N = 4$ gauge theory implies that the A-models with targets $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^L G, C)$ are equivalent provided the complexified Kähler classes are related by

$$(6) \quad {}^L \tau = -\frac{1}{n_g \tau}.$$

In mathematical terms, this means that the categories of A-branes attached to $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^L G, C)$ are equivalent. As explained in [12], one can understand this equivalence as coming from T-dualizing the Hitchin fibration of $\mathcal{M}_H(G, C)$. The Hitchin fibration is holomorphic in complex structure I , therefore this T-duality relates A-models in complex structure I for G and ${}^L G$. Note that the same T-duality induces an equivalence of B-models in complex structure I ; as explained in [11], this leads to the “baby” version of Geometric Langlands duality.

To make the statement of Montonen-Olive duality more useful, we would like to relate A-branes in complex structure I to some more familiar mathematical objects. As in [12], we can try to find a special coisotropic A-brane analogous to the canonical coisotropic A-brane of

[12]. For $\text{Re } \tau = 0$, this is the A-brane given by a rank-one line bundle on $\mathcal{M}_H(G, C)$ with a connection whose curvature is

$$F = \text{Im } \tau \omega_K.$$

This form is exact:

$$F = \text{Im } \tau \frac{1}{2\pi} \delta \int_C \text{Tr } \phi \wedge \delta A$$

We will call this brane the distinguished coisotropic A-brane, or d.c. brane for short, to distinguish it from the c.c. brane of [12]. The c.c. brane has $F = \text{Im } \tau \omega_J$. Note that the definition of the d.c. brane is independent of the complex structure on C , i.e. it is a topological object. On the other hand, the definition of the c.c. brane involves the complex structure on C in an essential way.

The space of boundary observables for the d.c. brane is the space of functions on $\mathcal{M}_H(G, C)$ holomorphic in complex structure $J = \omega_I^{-1} \omega_K$. In this complex structure the space $\mathcal{M}_H(G, C)$ can be identified with the moduli space of stable flat $G_{\mathbb{C}}$ connections on C , which we denote $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$. Furthermore, the algebra structure on the space of observables is noncommutative: it is the quantization of the algebra of holomorphic functions corresponding to the Poisson bivector

$$P = (\text{Im } \tau)^{-1} \Omega_J^{-1}.$$

where

$$\Omega_J = \omega_K + i\omega_I = -\frac{i}{4\pi} \int_C \text{Tr } \delta \mathcal{A} \wedge \delta \mathcal{A} = \frac{-i}{2\pi} \int_C dz d\bar{z} \text{Tr } \delta \mathcal{A}_z \wedge \delta \mathcal{A}_{\bar{z}}.$$

More precisely, the action of the A-model on a disc has the form

$$i \text{Im } \tau \int \phi^* \Omega_J$$

plus BRST exact terms. Thus $\text{Im } \tau$ plays the role of the inverse of the Planck constant. The algebra of boundary observables becomes noncommutative, and to first order in $(\text{Im } \tau)^{-1}$ we have

$$[f, g] = -i(\text{Im } \tau)^{-1} \Omega_J^{-1} (df, dg) + O((\text{Im } \tau)^{-2}),$$

where f and g are holomorphic functions on the moduli space of flat $G_{\mathbb{C}}$ -connections. The functions f and g may only be locally-defined, so we are really deforming the sheaf of holomorphic functions on $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$.

To any A-brane α in complex structure I we can attach a module over this noncommutative algebra. This module is the space of the states of the A-model on an interval with boundary conditions given

by α and the d.c. brane. More precisely, we can associate to an A-brane a sheaf of modules over the sheaf of boundary observables for the d.c. brane.

It has been argued in [10] (see also [15, 12, 9]) that whenever a coisotropic A-brane of maximal dimension exists, the category of A-branes is equivalent to the category of B-branes on a noncommutative deformation of the underlying complex manifold. In the present case, this complex manifold is the moduli space of stable flat $G_{\mathbb{C}}$ connections on C . Thus the Montonen-Olive duality can be reformulated as the statement that the derived categories of coherent sheaves on the quantizations of $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$ and $\mathcal{M}_{flat}({}^L G_{\mathbb{C}}, C)$ are equivalent. As usual, we expect that the equivalence is given by a certain object on the product of the quantized spaces. Its space of sections gives rise to a bi-module over the algebras of quantized functions on $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$ and $\mathcal{M}_{flat}({}^L G_{\mathbb{C}}, C)$, i.e. these algebras are derived-Morita-equivalent.

If $\text{Re } \tau \neq 0$, then there is a B-field on $\mathcal{M}_H(G, C)$ given by

$$B = -\text{Re } \tau \omega_I$$

If $\text{Re } \tau = n \in \mathbb{Z}$, then there exists an analogue of the d.c. brane which has the curvature

$$F = \text{Im } \tau \omega_I + \text{Re } \tau \omega_I.$$

This curvature 2-form is not exact: its cohomology class is n times the first Chern class of the determinant line bundle over $\mathcal{M}_H(G, C)$. The algebra of boundary observables for this brane is the same as for $\text{Re } \tau = 0$. Indeed, this is clear from the fact that the gauge theory for $\text{Re } \tau = n$ is isomorphic to the gauge theory for $\text{Re } \tau = 0$.

For more general values of $\text{Re } \tau$ there is no rank-one coisotropic A-brane on $\mathcal{M}_H(G, C)$. However, for rational values of $\text{Re } \tau$ there may exist distinguished coisotropic A-branes of higher rank. Indeed, the equation for the curvature of a $U(r)$ vector bundle on a coisotropic brane,

$$F = \text{Im } \tau \omega_K \cdot \mathbf{1} + \text{Re } \tau \omega_I \cdot \mathbf{1},$$

may admit solutions if $r \cdot \text{Re } \tau$ is integral. The space of boundary observables for such a brane is the space of holomorphic sections of a certain algebra bundle which is locally isomorphic to $\text{End}(\epsilon)$ for some rank- r holomorphic vector bundle ϵ on $\mathcal{M}_H(G, C)$. Such an algebra bundle is called an Azumaya algebra over $\mathcal{M}_H(G, C)$. The algebra of boundary observables is the quantization of the algebra of sections of this Azumaya algebra. While there is no canonical choice of such an Azumaya algebra, its category of modules depends only on $\text{Re } \tau$ (which determines the Morita-class of the Azumaya algebra). Presumably, this

remains true after quantization. Note however that rationality of $\text{Re } \tau$ is not preserved by the Montonen-Olive duality.

Let us discuss some examples of A-branes and their conjectural duals at $t = 0$. First of all, we have the d.c. brane itself. Since its curvature is of type $(1, 1)$ in complex structure K , it is a brane of type (A, A, B) in the notation of [12]. That is, it is an A-brane in complex structures I and J and is a B-brane with respect to complex structure K . A slightly more general (A, A, B) brane is obtained by twisting with a flat line bundle on $\mathcal{M}_H(G, C)$.

The other obvious example is the c.c. brane of [12], which is obtained by letting

$$F = \text{Im } \tau \omega_J.$$

This brane is of type (A, B, A) and depends on the complex structure on C . In fact, since the complex structures J and K are related by an isometry, it can be obtained from the d.c. brane by a hyperkähler rotation.

A large class of Lagrangian A-branes is obtained by considering all flat $G_{\mathbb{C}}$ -connections on C with a fixed $(0, 1)$ part, i.e. fixed holomorphic structure. This defines a topologically trivial Lagrangian submanifold of $\mathcal{M}_H(G, C)$ which is moreover a complex submanifold with respect to complex structure J . In other words, it is an A-brane of type (A, B, A) . For example, one can require the holomorphic structure to be trivial, or to require the flat connection to be an oper on C . Obviously, one may also consider all flat connections with a fixed $(1, 0)$ part. This condition also defines a complex Lagrangian submanifold with respect to complex structure J , i.e. it is a brane of type (A, B, A) .

Our final example of a Lagrangian submanifold in complex structure I is inspired by instanton Floer homology. Consider a 3-manifold N whose boundary is the Riemann surface C . We define $\alpha_N \subset \mathcal{M}_{flat}(G_{\mathbb{C}}, C)$ by saying that a point $p \in \mathcal{M}_{flat}(G_{\mathbb{C}}, C)$ belongs to α_N if and only if the corresponding flat connection is a restriction of a flat $G_{\mathbb{C}}$ -connection on N . It is well-known that α_N is a complex Lagrangian submanifold in $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$, i.e. it is a brane of type (A, B, A) . Note that the condition defining α_N is nonlocal on C . Thus while it is a valid A-brane for a 2d sigma-model with target $\mathcal{M}_H(G, C)$, it does not arise from a local boundary condition in 4d gauge theory.

Let us make some remarks on the action of Montonen-Olive duality on these A-branes. This duality maps branes of type (A, B, A) to branes of the same type. In particular, since the c.c. brane is flat along the fibers of the Hitchin fibration, its dual must be a Lagrangian (A, B, A) brane. Furthermore, it must intersect each nonsingular fiber

of the Hitchin fibration at a single point. It is plausible that the c.c. brane on $\mathcal{M}_H(G, C)$ is dual to the submanifold of opers in $\mathcal{M}_H({}^L G, C)$ [12]. Then the mirror of the d.c. brane is a Lagrangian (A, A, B) brane which is obtained from the oper brane by the hyperkähler rotation which turns J into K .

5. FROM NONCOMMUTATIVE B-BRANES TO TWISTED D-MODULES

We have argued above that for $\theta = 0$ the category of A-branes on the moduli space of Higgs bundles is equivalent to the category of B-branes on a noncommutative deformation of $\mathcal{M}_{flat}(G_{\mathbb{C}}, C)$. We now want to reinterpret this result in terms of twisted D-modules on $Bun_G(C)$.

This reinterpretation is based on the fact that the moduli stack of flat connections can be regarded as a twisted cotangent bundle over $Bun_G(C)$. This fact goes back to [1, 2].³ Let us recall what this means.

An affine bundle over a complex manifold M modeled on T^*M can be described as follows. Let q^i be local coordinate functions on a coordinate chart $U \subset M$, and \tilde{q}^i be local coordinate functions on a coordinate chart \tilde{U} . Let p_i and \tilde{p}_i be the corresponding “Darboux” coordinates on the fibers of T^*U and $T^*\tilde{U}$, respectively. The information about the affine bundle A is contained in how p_i is related to \tilde{p}_i on $U \cap \tilde{U}$:

$$(7) \quad \tilde{p}_i d\tilde{q}^i = p_i dq^i + \alpha_{U,\tilde{U}},$$

where $\alpha_{U,\tilde{U}}$ is a holomorphic 1-form on $U \cap \tilde{U}$. It is clear that the totality of these 1-forms is a 1-cocycle with values in the sheaf Ω_M^1 , and that cohomologous cocycles define isomorphic affine bundles. We will denote by $\alpha(A)$ the class in $H^1(\Omega^1)$ corresponding to the affine bundle A .

If $\alpha(A)$ is ∂ -closed, the 2-form $dp_i dq^i$ on the total space of A is well-defined and is holomorphic symplectic. In such a case one says that A is a twisted cotangent bundle over M . For example, if \mathcal{L} is a holomorphic line bundle on M equipped with a Hermitian metric, its curvature gives a ∂ -closed element in $H^1(\Omega_M^1)$. This cohomology class does not really depend on the choice of metric. We will denote this twisted cotangent bundle $A_{\mathcal{L}}(M)$.⁴ More generally, we can multiply the curvature of ∇

³Strictly speaking, the space relevant for us is the moduli space of stable flat connections, which is not a cotangent bundle. Rather, an open subset of $\mathcal{M}_{flat}(G, C)$ is a cotangent bundle to the moduli space of stable G -bundles on C . As a consequence, we will be really dealing with D-modules not on $Bun_G(C)$, but on the moduli space of stable G -bundles. We will gloss over this important subtlety.

⁴Another way to explain what $A_{\mathcal{L}}(M)$ is is to say that it is the space associated to the sheaf of holomorphic ∂ -connections on \mathcal{L} .

by a complex number λ ; the corresponding twisted cotangent bundle will be denoted $\mathbf{A}_{\mathcal{L}^\lambda}(M)$.

The basic fact we need is that the moduli space of flat $G_{\mathbb{C}}$ -connections on C equipped with the symplectic form Ω_J is isomorphic to the twisted cotangent bundle \mathbf{A} over $\mathrm{Bun}_G(C)$ such that

$$\alpha(\mathbf{A}) = \frac{i}{2\pi} \int_C dz d\bar{z} \mathrm{Tr} \delta A_z \wedge \delta A_{\bar{z}}.$$

The map to $\mathrm{Bun}_G(C)$ is the forgetful map which keeps only the $(0, 1)$ part of the connection. The fiber of the map is the space of ∂ -connections on a fixed holomorphic G -bundle, which is an affine space modeled on the cotangent space to $\mathrm{Bun}_G(C)$. Since the second cohomology of $\mathrm{Bun}_G(C)$ is one-dimensional and spanned by the class of the $(1, 1)$ form (5), the class α of the resulting twisted cotangent bundle \mathbf{A} must be proportional to (5). To fix the proportionality constant we note that the locally-defined ‘‘holomorphic symplectic potential’’ for the holomorphic symplectic form Ω_J is

$$-\frac{i}{2\pi} \int_C dz d\bar{z} \mathrm{Tr} \mathcal{A}_z \delta \mathcal{A}_{\bar{z}}.$$

We can make it globally well-defined by adding to it a locally-defined $(1, 0)$ form

$$-\frac{i}{2\pi} \int_C dz d\bar{z} \mathrm{Tr} (-A_z + i\phi_z) \delta \mathcal{A}_{\bar{z}}.$$

Acting on the resulting globally-defined $(1, 0)$ form with $\bar{\partial}$, we get a $(1, 1)$ form on $\mathrm{Bun}_G(C)$:

$$\frac{-i}{2\pi} \int_C dz d\bar{z} \mathrm{Tr} \delta \mathcal{A}_{\bar{z}}^\dagger \wedge \delta \mathcal{A}_{\bar{z}}.$$

This $(1, 1)$ form is cohomologous to

$$\frac{i}{2\pi} \int_C dz d\bar{z} \mathrm{Tr} \delta A_z \wedge \delta A_{\bar{z}}.$$

We can formulate this result in a slightly different way by noting that $\alpha(\mathbf{A})$ is $-2\pi i$ times the first Chern class of the line bundle \mathcal{L}_0 on $\mathrm{Bun}_G(C)$. Then we can identify the moduli space of flat connections with the twisted cotangent bundle $\mathbf{A}_{\mathcal{L}_0^{-1}}(\mathrm{Bun}_G)$.

The identification of the moduli space of flat $G_{\mathbb{C}}$ connections with a twisted cotangent bundle over $\mathrm{Bun}(G)$ enables one to quantize it in a straightforward way. Let M be a complex manifold, let \mathcal{L} be a holomorphic line bundle over it, and let λ be a nonzero complex

number. To quantize the structure sheaf of the complex symplectic manifold $\mathbf{A}_{\mathcal{L}^\lambda}(M)$ with the symplectic form

$$\frac{1}{\hbar} dp_j dq^j$$

we impose the commutation relations

$$[p_k, q^j] = -i\hbar \delta_k^j, \quad [p_k, p_j] = 0.$$

We can try to solve these commutation relations by letting

$$p_j = -i\hbar \nabla_j,$$

where ∇_j is a (locally-defined) holomorphic differential on a holomorphic line bundle \mathcal{L}_\hbar (or some complex power of a holomorphic line bundle). Then the quantization of sheaf of holomorphic functions can be identified with the sheaf of holomorphic differential operators on \mathcal{L}_\hbar . The line bundle \mathcal{L}_\hbar is fixed by imposing the transformation law (7). This implies $\mathcal{L}_\hbar = \mathcal{L}^{i\lambda/\hbar}$. Note that the exponent $i\lambda/\hbar$ may be complex.

At this stage a subtlety creeps in. The first Chern class of \mathcal{L}_\hbar that we got is of order $1/\hbar$; this happened because the transformation law for p_j was nontrivial already at the classical level. One may wonder if there are quantum corrections to the transformation law and therefore to \mathcal{L}_\hbar . In fact, as explained in [12], the most natural quantization recipe leads to such a correction. In the case of trivial \mathcal{L} it is natural to take $\mathcal{L}_\hbar = K^{1/2}$, where K is the canonical line bundle of M . This choice ensures that the algebra of twisted differential operators is isomorphic to its opposite, something which is required by a discrete symmetry $p \rightarrow -p$ present for trivial \mathcal{L} . From our point of view, $K^{1/2}$ represents a correction to the first Chern class of \mathcal{L}_\hbar which is of order \hbar^0 . Assuming that there are no higher-order corrections, we conclude that the quantization of $\mathbf{A}_{\mathcal{L}}(M)$ is the sheaf of holomorphic differential operators on $K^{1/2} \otimes \mathcal{L}^{i\lambda/\hbar}$.

In our case $M = \mathrm{Bun}_G$, $\mathcal{L} = \mathcal{L}_0$, $\lambda = -1$, and $\hbar = (\mathrm{Im} \tau)^{-1}$. We conclude that the sheaf of observables for the d.c. brane can be identified with the sheaf of holomorphic differential operators on $K^{1/2} \otimes \mathcal{L}_0^{-i\mathrm{Im}\tau} = \mathcal{L}_0^{-h^\vee - i\mathrm{Im}\tau}$. Therefore the Montonen-Olive duality at $t = 0$ implies the statement of the quantum geometric Langlands duality for $\mathbf{q} = -i\mathrm{Im}\tau$. The value of \mathbf{q} that we get in this way is purely imaginary.

6. LINE AND SURFACE OPERATORS AT $t = 0$

6.1. Line operators. There are several different kinds of line operators. One kind of a line operator is built using the descent procedure [17]. Given a local BRST-invariant 0-form \mathcal{O} , one may consider its descendant 1-form defined by

$$d\mathcal{O} = \delta\mathcal{O}^{(1)}$$

The integral of $\mathcal{O}^{(1)}$ along a cycle γ is then BRST-invariant. For example, at $t = 0$ we may take $\mathcal{O} = \text{Tr } \sigma^2$, then

$$\mathcal{O}^{(1)} = \text{Tr } \psi \sigma.$$

Another kind of line operator is a Wilson-type line operator, i.e. the holonomy of some connection. This connection must be BRST-invariant up to a gauge transformation. At $t = 0$ there are no observables of this kind.

Finally, one may try to construct a disorder line operator. In the present case, one needs to look for singular gauge field configurations which solve the BPS equations. If we assume that near the singularity the solutions become abelian, spherically symmetric and scale-invariant, we can easily see that no disorder operators can be constructed. Indeed, if the disorder operator is inserted at $x^i = 0, i = 1, 2, 3$, then the vanishing of the BRST variation of χ^+ requires

$$F_{ij} = \epsilon_{ijk} F_{4k}.$$

Spherical symmetry and scale-invariance require

$$F_{ij} = \frac{\rho}{2} \epsilon_{ijk} \frac{x^k}{r^3},$$

where $\rho \in \mathfrak{t}$ lies in the cocharacter lattice of G (this is the Goddard-Nuyts-Olive quantization condition [4]). But then the “electric field” F_{4k} is given by

$$F_{4k} = \frac{\rho}{2} \frac{x^k}{r^3}.$$

This is a Coulomb field, but the electric charge corresponding to it is not quantized properly and in fact is purely imaginary rather than real. Indeed, if we pass to Minkowski signature by replacing $\partial_4 \rightarrow -i\partial_0$ and $F_{4k} \rightarrow -iF_{0k}$, we find

$$F_{0k} = \frac{i\rho}{2} \frac{x^k}{r^3}.$$

Thus such a singularity does not correspond to a gauge-invariant line operator.

6.2. Surface operators. Surface operators defined in [7] work equally well for all t , including $t = 0$ and $t = \infty$. The most basic surface operators are parameterized by a quadruple $(\alpha, \beta, \gamma, \eta)$, where β, γ take values in the Cartan subalgebra \mathfrak{t} of G , α takes values in the maximal torus \mathbb{T} of G , and η takes values in the maximal torus ${}^L\mathbb{T}$ of ${}^L G$. In addition, one identifies quadruples which differ by an element of the Weyl group and requires the quadruple $(\alpha, \beta, \gamma, \eta)$ to be regular, in the sense that no nontrivial element of the Weyl group leaves it invariant. More generally, one can consider surface operators associated with a Levi subgroup of G [7]; the basic ones correspond to taking \mathbb{T} as the Levi subgroup.

At $t = 0$ the parameters β and γ affect only the complex structure of $\mathcal{M}_H(G, C)$ and therefore locally are irrelevant as far as the A-model is concerned.⁵ The important parameters are α and η since they control the Kähler form and the B-field on $\mathcal{M}_H(G, C)$. Montonen-Olive duality acts on them by

$$(\alpha, \eta) \mapsto ({}^L\alpha, {}^L\eta) = (\eta, -\alpha).$$

We note in passing that although there are no nontrivial “bulk” line operators at $t = 0$, line operators living on surface operators seem to exist in some cases. For example, as discussed in [7], a noncontractible loop in the space of “irrelevant” parameters β and γ should correspond to such a line operator. Such noncontractible loops exist if the pair (α, η) is not regular, i.e. preserved by some nontrivial element of the Weyl group.

6.3. Ramified Quantum Geometric Langlands. Surface operators affect the QGL story in the following way. Suppose there are surface operators inserted at points $p_1, \dots, p_s \in C$. Then the effective 2d theory at $t = 0$ is an A-model whose target is the moduli space of stable parabolic Higgs bundles on C , with parabolic structure at p_1, \dots, p_s . We denote this moduli space $\mathcal{M}_H(G, C; p_1, \dots, p_s)$. The Kähler form ω_I of $\mathcal{M}_H(G, C; p_1, \dots, p_s)$ has periods which depend linearly on $\text{Im } \tau$ and $\alpha_1, \dots, \alpha_s$; the B-field has periods which depend linearly on $\text{Re } \tau$ and η_1, \dots, η_s . More precisely, as explained in [7], the second cohomology of $\mathcal{M}_H(G, C)$ can be naturally identified with

$$\mathbb{Z} \oplus \bigoplus_{k=1}^s \Lambda_{char}(G),$$

⁵Varying $\beta + i\gamma$ along a closed loop in the allowed parameter space induces an autoequivalence of the category of A-branes; these autoequivalences play an important role in [7].

where $\Lambda_{char}(G) = \text{Hom}(\mathbb{T}, U(1)) \subset {}^L\mathbf{t}$ is the character lattice of G . In terms of this identification the cohomology class of $\omega_I/2\pi$ is given by

$$\left[\frac{\omega_I}{2\pi} \right] = e \oplus (- \oplus_{k=1}^s \alpha_k^*),$$

where the star denotes an isomorphism of \mathbf{t} and ${}^L\mathbf{t}$ coming from the Killing form $-\text{Tr}$. The Kähler form is $\text{Im } \tau \omega_I$. The cohomology class of the B-field is given by [7]

$$\left[\frac{B}{2\pi} \right] = (-\text{Re } \tau) \oplus \oplus_{k=1}^s \eta_k.$$

Montonen-Olive duality now says that the categories of A-branes on $\mathcal{M}_H(G, C; p_1, \dots, p_s)$ and $\mathcal{M}_H({}^L G, C; p_1, \dots, p_s)$ are equivalent if τ and ${}^L\tau$ are related as in (6) and the parameters (α, η) and $({}^L\alpha, {}^L\eta)$ are related by

$$(8) \quad {}^L\alpha_k = \eta_k, \quad {}^L\eta_k = -\alpha_k, \quad k = 1, \dots, s.$$

To connect this statement to a ramified version of QGL, we consider the special case $\text{Re } \tau = 0$. We can try to define a d.c. brane by requiring

$$F + B = \text{Im } \tau \omega_K$$

for some cohomologically trivial 2-form F . This ensures that we can interpret the 2-form F as the curvature of a trivial line bundle on $\mathcal{M}_H(G, C)$. As explained in [7], the cohomology class of ω_K is given by

$$\left[\frac{\omega_K}{2\pi} \right] = 0 \oplus (- \oplus_{k=1}^s \gamma_k^*).$$

Therefore the “irrelevant” parameters γ_k are determined by η_k :

$$\gamma_k = -(\text{Im } \tau)^{-1} \eta_k^*.$$

This restriction is immaterial since the category of A-branes does not depend on γ anyway; we just need it to be able to construct a d.c. brane.

Classically, the algebra of boundary observables for the d.c. brane is the algebra of holomorphic functions on $\mathcal{M}_H(G, C; p_1, \dots, p_s)$ in complex structure J . The latter space can be identified with the moduli space of stable parabolic $G_{\mathbb{C}}$ local systems on C .

On the quantum level we need to quantize this algebra, or rather the corresponding sheaf of algebras. As before, this is facilitated by thinking about the moduli space of parabolic local systems as a twisted cotangent bundle to the moduli space of parabolic G -bundles on C . The natural holomorphic symplectic form $\Omega_J = \omega_K + i\omega_I$ becomes the canonical symplectic form on the twisted cotangent bundle. Therefore the quantized sheaf of boundary observables is the sheaf of holomorphic

differential operators on $K^{1/2} \otimes \mathcal{L}_\hbar$, where a priori \mathcal{L}_\hbar is a product of complex powers of line bundles over $\text{Bun}_G(C; p_1, \dots, p_s)$. The first Chern class of \mathcal{L}_\hbar is the cohomology class of $\text{Im } \tau \Omega_J$ divided by -2π :

$$c_1(\mathcal{L}_\hbar) = i \text{Im } \tau ((-e) \oplus \bigoplus_{k=1}^s (\alpha_k^* + i(\text{Im } \tau)^{-1} \eta_k)).$$

This class is no longer purely imaginary, but in the classical regime (large $\text{Im } \tau$) its imaginary part is much larger than the real one. Note also that this class is a holomorphic function of $\alpha_k^* + i(\text{Im } \tau)^{-1} \eta_k$, as expected for the A-model in complex structure I . The Montonen-Olive duality suggests that the derived categories of twisted D-modules on $\text{Bun}_G(C; p_1, \dots, p_s)$ and $\text{Bun}_{^L G}(C; p_1, \dots, p_s)$ are equivalent if α_k, η_k and ${}^L \alpha_k$ and ${}^L \eta_k$ are related by (8) and $\text{Im } \tau$ and $\text{Im } {}^L \tau$ are related by

$$\text{Im } {}^L \tau = \frac{1}{n_g \text{Im } \tau}$$

This is a ramified version of the Quantum Geometric Langlands duality.

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